JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **29**, No. 1, 2016
http://dx.doi.org/10.14403/jcms.2016.29.1.103

STABILITY OF AN ADDITIVE FUNCTIONAL INEQUALITY IN BANACH SPACES

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Abstract. In this paper, we prove the generalized Hyers-Ulam stability of the additive functional inequality

$$\|f(x_1+x_2)+f(x_2+x_3)+\cdots+f(x_n+x_1)\| \leq \|tf(x_1+x_2+\cdots+x_n)\|$$

in Banach spaces where a positive integer $n \geq 3$ and a real number t such that $2 \leq t < n$.

1. Introduction and preliminaries

In 1940, S.M. Ulam [5] suggested the stability problem of functional equations concerning the stability of group homomorphisms as follows: *Let $(\mathcal{G}, \circ)$ be a group and let $(\mathcal{H}, \star, d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta = \delta(\varepsilon) > 0$ such that if a mapping $f : \mathcal{G} \to \mathcal{H}$ satisfies the inequality $d\big(f(x \circ y), f(x) \star f(y)\big) < \delta$ for all $x, y \in \mathcal{G}$, then a homomorphism $F : \mathcal{G} \to \mathcal{H}$ exits with $d\big(f(x), F(x)\big) < \varepsilon$ for all $x \in \mathcal{G}$?*

In the next year, D.H. Hyers [2] gave a first (partial) affirmative answer to the question of Ulam for Banach spaces as follows: *If $\delta > 0$ and if $f : \mathcal{E} \to \mathcal{F}$ is a mapping between Banach spaces $\mathcal{E}$ and $\mathcal{F}$ satisfying*

$$\big\|f(x+y) - f(x) - f(y)\big\| \leq \delta$$

for all $x, y \in \mathcal{E}$, then there is a unique additive mapping $A : \mathcal{E} \to \mathcal{F}$ such that $\big\|f(x) - A(x)\big\| \leq \delta$ for all $x, y \in \mathcal{E}$.

Thereafter, we call that type the Hyers-Ulam stability.

Received December 30, 2015; Accepted February 05, 2016.
2010 Mathematics Subject Classification: Primary 39B82.
Key words and phrases: additive functional inequality, Banach space.

2. Hyers-Ulam stability in Banach spaces

Throughout this paper, let $\mathcal{X}$ be a normed linear space and $\mathcal{Y}$ a Banach space. In 2007, C. Park, Y. S. Cho and M.-H. Han [4] proved the generalized Hyers-Ulam stability of the additive functional inequality

$$\big\|f(x)+f(y)+f(z)\big\| \le \big\|f(x+y+z)\big\|$$

in Banach spaces. In 2013, S.-C Chung [1] prove the generalized Hyers-Ulam stability of the additive functional inequality

$$\|f(2x_1)+f(2x_2)+\cdots+f(2x_n)\| \le \|tf(x_1+x_2+\cdots+x_n)\|$$

in Banach spaces where a positive integer $n \ge 3$ and a real number t such that $2 \le t < n$.

In this paper, we prove the generalized Hyers-Ulam stability of the additive functional inequality

$$\|f(x_1+x_2)+f(x_2+x_3)+\cdots+f(x_n+x_1)\| \le \|tf(x_1+x_2+\cdots+x_n)\|$$

in Banach spaces.

Lemma 2.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping. For an odd integer n and a real number t suppose that $3 \le n$ and $2 \le t < n$. Then it is additive if and only if it satisfies*

$$\|f(x_1+x_2)+f(x_2+x_3)+\cdots+f(x_n+x_1)\| \le \|tf(x_1+x_2+\cdots+x_n)\| \tag{2.1}$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{X}$.

Proof. If f is additive, then clearly

$$\begin{aligned}
&\big\|f(x_1+x_2)+f(x_2+x_3)+\cdots+f(x_n+x_1)\big\| \\
&= \|2f(x_1+x_2+\cdots+x_n)\| \\
&\le \|tf(x_1+x_2+\cdots+x_n)\|
\end{aligned}$$

for all $x_i \in \mathcal{X}$.

Conversely assume that f satisfies (2.1). Letting $x_i = 0 (1 \le i \le n)$ in (2.1), we have $\big\|nf(0)\big\| \le \big\|tf(0)\big\|$ and so $f(0) = 0$ by the hypothesis. Putting $x_1 = x, x_2 = -x, x_i = 0 (3 \le i \le n)$ in (2.1), we get $\big\|f(-x) + f(x)\big\| \le \big\|tf(0)\big\| = 0$ and so $f(-x) = -f(x)$ for all $x \in \mathcal{X}$. Setting $x_1 = x, x_i = (-1)^i y (2 \le i \le n-1), x_n = -x-y$ in (2.1), we have

$$\big\|f(x+y)+f(-x)+f(-y)\big\| \le \big\|tf(0)\big\| = 0$$

for all $x, y \in \mathcal{X}$. Thus we obtain $f(x+y) = f(x)+f(y)$ for all $x, y \in \mathcal{X}$. $\square$

THEOREM 2.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping with $f(0) = 0$. For an odd integer n and a real number t suppose that $3 \leq n$ and $2 \leq t < n$. If there is a function $\varphi : \mathcal{X}^n \to [0, \infty)$ satisfying*

$$\begin{aligned} &\|f(x_1 + x_2) + f(x_2 + x_3) + \cdots + f(x_n + x_1)\| \\ &\leq \|tf(x_1 + x_2 + \cdots + x_n)\| + \varphi\big(x_1, x_2, \cdots, x_n\big) \end{aligned} \tag{2.2}$$

and

$$\widetilde{\varphi}(x_1, x_2, \cdots, x_n) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi\big((-2)^j x_1, (-2)^j x_2, \cdots, (-2)^j x_n\big) < \infty \tag{2.3}$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{X}$, then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that for all $x \in \mathcal{X}$

$$\big\|f(x) - A(x)\big\| \leq \frac{1}{2} \widetilde{\varphi}(-x, x_2, \cdots, x_{n-1}, 2x) \tag{2.4}$$

where $x_i = (-1)^{i-1} x (2 \leq i \leq n-1)$.

Proof. Putting $x_1 = (-2)^l(-x), x_i = (-2)^l(-1)^{i-1}x(2 \leq i \leq l-1), x_n = (-2)^{l+1}(-x)$, respectively, and dividing by 2^{l+1} in (2.2), since $f(0) = 0$, we get

$$\begin{aligned} &\left\| \frac{f\big((-2)^{l+1}x\big)}{(-2)^{l+1}} - \frac{f\big((-2)^l x\big)}{(-2)^l} \right\| \\ &\leq \frac{1}{2^{l+1}} \varphi\big((-2)^l(-x), (-2)^l(-x), (-2)^l x, \cdots, (-2)^l(-x), (-2)^{l+1}(-x)\big) \end{aligned}$$

for all $x \in \mathcal{X}$ and all nonnegative integers l. From the above inequality, we have

$$\begin{aligned} &\left\| \frac{f\big((-2)^l x\big)}{(-2)^l} - \frac{f\big((-2)^m x\big)}{(-2)^m} \right\| \leq \sum_{j=m}^{l-1} \left\| \frac{f\big((-2)^{j+1}x\big)}{(-2)^{j+1}} - \frac{f\big((-2)^j x\big)}{(-2)^j} \right\| \\ &\leq \sum_{j=m}^{l-1} \frac{1}{2^{j+1}} \varphi\big((-2)^j(-x), (-2)^j(-x), (-2)^j x, \cdots, (-2)^j(-x), (-2)^{j+1}(-x)\big) \end{aligned} \tag{2.5}$$

for all $x \in \mathcal{X}$ and all nonnegative integers m, l with $m < l$. By the condition (2.3), the sequence $\left\{ \frac{f((-2)^l x)}{(-2)^l} \right\}$ is a Cauchy sequence for all $x \in \mathcal{X}$. Since $\mathcal{Y}$ is complete, the sequence $\left\{ \frac{f((-2)^l x)}{(-2)^l} \right\}$ converges for all $x \in \mathcal{X}$. So one can define a mapping $A : \mathcal{X} \to \mathcal{Y}$ by $A(x) :=$

$\lim_{l\to\infty} \frac{f\left((-2)^l x\right)}{(-2)^l}$ for all $x \in \mathcal{X}$. Taking $m = 0$ and letting n tend to ∞ in (2.5), we have the inequality (2.4).

Replacing $x_i (1 \le i \le n)$ by $(-2)^l x_i$, respectively, and dividing by 2^l in (2.2), we obtain

$$\left\| \frac{f\left((-2)^l (x_1 + x_2)\right)}{(-2)^l} + \frac{f\left((-2)^l (x_2 + x_3)\right)}{(-2)^l} + \cdots + \frac{f\left((-2)^l (x_n + x_1)\right)}{(-2)^l} \right\|$$
$$\le \left\| \frac{tf\left((-2)^l (x_1 + x_2 + \cdots + x_n)\right)}{(-2)^l} \right\| + \frac{1}{2^l}\varphi\left((-2)^l x_1, (-2)^l x_2, \cdots, (-2)^l x_n\right)$$

for all $x_i \in \mathcal{X}$ and all nonnegative integers l. Since (2.3) gives that

$$\lim_{l\to\infty} \frac{1}{2^l}\varphi\left((-2)^l x_1, (-2)^l x_2, \cdots, (-2)^l x_n\right) = 0$$

for all $x_i \in \mathcal{X}$, letting l tend to ∞ in the above inequality, we have

$$\left\|A(x_1 + x_2) + A(x_2 + x_3) + \cdots + A(x_n + x_1)\right\| \le \left\|tA(x_1 + x_2 + \cdots + x_n)\right\|.$$

So by Lemma 2.1 A is an additive mapping.

Let $A' : \mathcal{X} \to \mathcal{Y}$ be another additive mapping satisfying (2.4). Since both A and A' are additive, we have

$$\left\|A(x) - A'(x)\right\| = \frac{1}{2^l}\left\|A\left((-2)^l x\right) - A'\left((-2)^l x\right)\right\|$$
$$\le \frac{1}{2^l}\left(\left\|A\left((-2)^l x\right) - f\left((-2)^l x\right)\right\| + \left\|f\left((-2)^l x\right) - A'\left((-2)^l x\right)\right\|\right)$$
$$\le \frac{1}{2^l}\widetilde{\varphi}\left((-2)^l(-x), (-2)^l(-x), (-2)^l x, \cdots, (-2)^l(-x), (-2)^{l+1}(-x)\right)$$
$$= \sum_{j=l}^{\infty} \frac{1}{2^j}\varphi\left((-2)^l(-x), (-2)^l(-x), (-2)^l x, \cdots, (-2)^l(-x), (-2)^{l+1}(-x)\right)$$

which goes to zero as $l \to \infty$ for all $x \in \mathcal{X}$ by (2.3). Therefore, A is a unique additive mapping satisfying (2.4), as desired. $\square$

Theorem 2.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping with $f(0) = 0$. If there is a function $\varphi : \mathcal{X}^n \to [0, \infty)$ satisfying (2.2) and*

$$\text{(2.6)} \quad \widetilde{\varphi}(x_1, x_2, \cdots, x_n) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x_1}{(-2)^j}, \frac{x_2}{(-2)^j}, \cdots, \frac{x_n}{(-2)^j}\right) < \infty$$

for all $x_i \in \mathcal{X}$, then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\text{(2.7)} \qquad \left\|f(x) - A(x)\right\| \le \frac{1}{2}\widetilde{\varphi}(-x, x_2, \cdots, x_{n-1}, 2x)$$

where $x_i = (-1)^{i-1}x (2 \le i \le n-1)$.

Proof. Putting $x_1 = \frac{-x}{(-2)^l}, x_i = (-1)^i \frac{-x}{(-2)^l} (2 \le i \le n-1), x_n = \frac{-x}{(-2)^{l-1}}$, respectively, and multiplying by 2^{l-1} in (2.2), since $f(0) = 0$, we have

$$\begin{aligned}&\left\|(-2)^l f\left(\frac{x}{(-2)^l}\right) - (-2)^{l-1} f\left(\frac{x}{(-2)^{l-1}}\right)\right\|\\ &\le 2^{l-1}\varphi\left(\frac{-x}{(-2)^l}, \frac{-x}{(-2)^l}, \frac{x}{(-2)^l}, \cdots, \frac{-x}{(-2)^l}, \frac{-x}{(-2)^{l-1}}\right)\end{aligned}$$

for all $x \in \mathcal{X}$ and all $l \in \mathbb{N}$. From the above inequality, we get

$$\begin{aligned}(2.8)\qquad &\left\|(-2)^l f\left(\frac{x}{(-2)^l}\right) - (-2)^m f\left(\frac{x}{(-2)^m}\right)\right\|\\ &\le \sum_{j=m+1}^{l} \left\|(-2)^j f\left(\frac{x}{(-2)^j}\right) - (-2)^{j-1} f\left(\frac{x}{(-2)^{j-1}}\right)\right\|\\ &\le \sum_{j=m+1}^{l} 2^{j-1}\varphi\left(\frac{-x}{(-2)^j}, \frac{-x}{(-2)^j}, \frac{x}{(-2)^j}, \cdots, \frac{-x}{(-2)^j}, \frac{-x}{(-2)^{j-1}}\right)\end{aligned}$$

for all $x \in \mathcal{X}$ and all nonnegative integers m, l with $m < l$. From (2.6), the sequence $\left\{(-2)^l f\left(\frac{x}{(-2)^l}\right)\right\}$ is a Cauchy sequence for all $x \in \mathcal{X}$. Since $\mathcal{Y}$ is complete, the sequence $\left\{(-2)^l f\left(\frac{x}{(-2)^l}\right)\right\}$ converges for all $x \in \mathcal{X}$. So one can define a mapping $A : \mathcal{X} \to \mathcal{Y}$ by $A(x) := \lim_{l\to\infty} (-2)^l f\left(\frac{x}{(-2)^l}\right)$ for all $x \in \mathcal{X}$. To prove that A satisfies (2.7), putting $m = 0$ and letting $n \to \infty$ in (2.8), we have

$$\begin{aligned}\|f(x) - A(x)\| &\le \sum_{j=1}^{\infty} 2^{j-1}\varphi\left(\frac{-x}{(-2)^j}, \frac{-x}{(-2)^j}, \frac{x}{(-2)^j}, \cdots, \frac{-x}{(-2)^j}, \frac{-x}{(-2)^{j-1}}\right)\\ &= \frac{1}{2}\widetilde{\varphi}(-x, -x, x, \ldots, -x, 2x)\end{aligned}$$

for all $x \in \mathcal{X}$.

Replacing $x_i (1 \le i \le n)$ by $\frac{x_i}{(-2)^l}$, respectively, and multiplying by 2^l in (2.2), we obtain

$$\begin{aligned}&\left\|(-2)^l f\left(\frac{x_1+x_2}{(-2)^l}\right)+(-2)^l f\left(\frac{x_2+x_3}{(-2)^l}\right)+\cdots+(-2)^l f\left(\frac{x_n+x_1}{(-2)^l}\right)\right\|\\ \le &\left\|t(-2)^l f\left(\frac{x_1+x_2+\cdots+x_n}{(-2)^l}\right)\right\|+2^l\varphi\left(\frac{x_1}{(-2)^l},\frac{x_2}{(-2)^l},\cdots,\frac{x_n}{(-2)^l}\right)\end{aligned}$$

for all $x_i \in \mathcal{X}$ and all nonnegative integers l. From (2.6) we have the following

$$\lim_{l\to\infty} 2^l\varphi\left(\frac{x_1}{(-2)^l},\frac{x_2}{(-2)^l},\cdots,\frac{x_n}{(-2)^l}\right)=0$$

for all $x_i \in \mathcal{X}$, if we let $l \to \infty$ in the above inequality, then we have

$$\left\|A(x_1+x_2)+A(x_2+x_3)+\cdots+A(x_n+x_1)\right\| \le \left\|tA(x_1+x_2+\cdots+x_n)\right\|.$$

for all $x_i \in \mathcal{X}$. By Lemma 2.1, the mapping A is additive. The rest of the proof is similar to the corresponding part of the proof of Theorem 2.2. □

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